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GLOBAL SOLUTIONS FOR THE ROTATING NAVIER-STOKES EQUATIONS

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1. Introduction

This paper is survey of the paper [14]. We consider the Cauchy problems for the Navier-Stokes equations with the Coriolis force

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + \Omega e_3 \times u + (u \cdot \nabla) u + \nabla p = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \mathrm{div}\ u = 0 & t > 0,\ x \in \mathbb{R}^3, \\ u(0, x) = u_0(x) & x \in \mathbb{R}^3, \end{cases} \tag{NSC}$$

where $u = u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$ and $p = p(t,x)$ denote the unknown velocity field and the unknown pressure of the fluid at the point $(t,x) \in (0,\infty) \times \mathbb{R}^3$, respectively, while $u_0 = u_0(x) = (u_{0,1}(x), u_{0,2}(x), u_{0,3}(x))$ denotes the given initial velocity field satisfying the compatibility condition div $u_0 = 0$. Here, $\Omega \in \mathbb{R}$ is the speed of rotation around the vertical unit vector $e_3 = (0,0,1)$.

For (NSC) with $\Omega = 0$, there are a lot of results for the existence of global solutions. It is known that global smooth solutions are obtained for small initial data in some scaling invariant function spaces. Kato [8] studied the existence of global solutions for small initial data in the Lebesgue space $L^3(\mathbb{R}^3)$. Here, the space $L^3(\mathbb{R}^3)$ is scaling invariant to the equation (NSC) with $\Omega = 0$. In fact, for a solution u let u_λ be defined by $u_\lambda(t,x) := \lambda u(\lambda^2 t, \lambda x)$ for $\lambda > 0$. Then, u_λ is also a solution to (NSC) with $\Omega = 0$ and we have the following norm invariant property in the Lebesgue space:

$$\|u_\lambda(0)\|_{L^p(\mathbb{R}^3)} = \|u(0)\|_{L^p(\mathbb{R}^3)} \text{ for any } \lambda > 0 \quad \text{if } p = 3.$$

On the results for small initial data in such scaling invariant function spaces, Kozono-Yamazaki [20] studied in the Besov spaces $\dot{B}_{p,\infty}^{-1+\frac{n}{p}}(\mathbb{R}^3)$ with $3 < p < \infty$, Koch-Tataru [18] studied in the class of bounded mean oscillation $BMO^{-1}(\mathbb{R}^3)$.

In this paper, we take the speed $|\Omega|$ large and show the existence of global solutions to (NSC) for large initial data in $\dot{H}^s(\mathbb{R}^3)$ with $s \geq 1/2$. In particular, we give a sufficient condition on the norm of initial data and the speed Ω for the existence of global solutions. For the existence of global solutions to (NSC), Chemin-Desjardins-Gallagher-Grenier [6, 7] proved that for any initial data $u_0 \in L^2(\mathbb{R}^2)^2 + H^{\frac{1}{2}}(\mathbb{R}^3)^3$, there exists a positive parameter Ω_0 such that for every $\Omega \in \mathbb{R}$ with $|\Omega| \geq \Omega_0$ there exists a unique global solution. Babin-Mahalov-Nicolaenko [2, 3, 4] showed the existence of global solutions and the regularity of the solutions to (NSC) for the periodic initial data with large $|\Omega|$. On the other hand, Giga-Inui-Mahalov-Saal [11] showed the existence of global solutions for small initial data $u_0 \in FM_0^{-1}(\mathbb{R}^3)$ which smallness is independent of $\Omega \in \mathbb{R}$. On such other results of global solutions for small initial data, Hieber-Shibata [12] studied in the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$, Konieczny-Yoneda [19] studied in the Fourier-Besov space $F\dot{B}^{2-\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ with $1 < p \le \infty$. We note that the spaces $FM_0^{-1}(\mathbb{R}^3)$, $H^{\frac{1}{2}}(\mathbb{R}^3)$ and $F\dot{B}^{2-\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)$ are scaling invariant spaces to (NSC) with $\Omega = 0$.

We consider the following integral equation:

$$u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)d\tau, \tag{IE}$$

where $\mathbb{P} = (\delta_{ij} + R_iR_j)_{1\le i,j\le 3}$ denotes the Helmholtz projection onto the divergence-free vector fields and $T_\Omega(\cdot)$ denotes the semigroup corresponding to the linear problem of (NSC), which is given explicitly by

$$T_\Omega(t)f = \mathcal{F}^{-1}\Big[\cos\Big(\Omega\frac{\xi_3}{|\xi|}t\Big)e^{-t|\xi|^2}I\widehat{f}(\xi) + \sin\Big(\Omega\frac{\xi_3}{|\xi|}t\Big)e^{-t|\xi|^2}R(\xi)\widehat{f}(\xi)\Big]$$

for $t \ge 0$ and divergence-free vector fields f. Here, I is the identity matrix in $\mathbb{R}^3$, R_j $(j = 1,2,3)$ is the Riesz transform and $R(\xi)$ is the skew-symmetric matrix symbol related to the Riesz transform, which is defined by

$$R(\xi) := \frac{1}{|\xi|}\begin{pmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{pmatrix} \quad \text{for } \xi \in \mathbb{R}^3\setminus\{0\}.$$

We refer to Babin-Mahalov-Nikolaenko [1, 2, 3], Giga-Inui-Mahalov-Saal [10] and Hieber-Shibata [12] for the derivation of the explicit form of $T_\Omega(\cdot)$.

We consider the initial data $u_0 \in \dot{H}^s(\mathbb{R}^3)$ with $1/2 \le s < 3/4$ to establish the existence theorem on global solutions. In the case $s > 1/2$, the sufficient speed Ω is characterized by the norm of initial data $\|u_0\|_{\dot{H}^s}$. In the case $s = 1/2$, the sufficient speed Ω is characterized by each precompact set $K \subset \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$, which the initial data belongs to. Our theorem for $s > 1/2$ is the following.

Theorem 1.1. *Let $\Omega \in \mathbb{R}\setminus\{0\}$, and let s, p and θ satisfy*

$$\frac{1}{2} < s < \frac{3}{4}, \quad \frac{1}{3} + \frac{s}{9} < \frac{1}{p} < \frac{2}{3} - \frac{s}{3}, \tag{1.1}$$

$$\frac{s}{2} - \frac{1}{2p} < \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}, \quad \frac{3}{4} - \frac{3}{2p} \le \frac{1}{\theta} < 1 - \frac{2}{p}. \tag{1.2}$$

Then, there exists a positive constant $C = C(s,p,\theta) > 0$ such that for any initial velocity field $u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ with

$$\|u_0\|_{\dot{H}^s} \le C|\Omega|^{\frac{s}{2}-\frac{1}{4}} \quad \textit{and} \quad \operatorname{div} u_0 = 0, \tag{1.3}$$

there exists a unique global solution $u \in C([0,\infty), \dot{H}^s(\mathbb{R}^3))^3 \cap L^\theta(0,\infty; \dot{H}^s_p(\mathbb{R}^3))^3$ to (NSC).

Remark 1.2. The existence of global solutions for small initial data $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$ were shown by Hieber-Shibata [12]. The size condition (1.3) on initial data can be regarded as a continuous extension of that in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$. Indeed, Hieber-Shibata [12] assumed the smallness condition $\|u_0\|_{\dot{H}^{\frac{1}{2}}} \le \delta$ for some $\delta > 0$, which corresponds to our condition (1.3) with $s = 1/2$.

Remark 1.3. The space $L^{\theta_0}(0,\infty;\dot{H}^{s_0}_{p_0}(\mathbb{R}^3))$ is scaling invariant to (NSC) in the case $\Omega = 0$ if θ_0, s_0 and p_0 satisfy

$$\frac{2}{\theta_0} + \frac{3}{p_0} = 1 + s_0. \tag{1.4}$$

On the first condition of (1.2), we see that

$$\frac{2}{\theta} + \frac{3}{p} < \frac{5}{4} + \frac{s}{2} < 1 + s \quad \text{if } s > \frac{1}{2}.$$

Therefore, the space $L^{\theta}(0,\infty;\dot{H}^{s}_{p}(\mathbb{R}^3))$ in Theorem 1.1 includes more regular functions than those in the scaling invariant spaces.

In the case $s = 1/2$, it seems difficult to obtain the sufficient condition on the size of initial data and the speed for the existence of global solutions since $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ is scaling invariant to (NSC) with $\Omega = 0$. Then, we introduce precompact sets in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ to obtain the following result.

Theorem 1.4. *Let K be an arbitrary precompact set in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)^3$. Then, there exists $\omega(K) > 0$ such that for any $\Omega \in \mathbb{R}$ with $|\Omega| > \omega(K)$ and for any $u_0 \in K$ with* $\operatorname{div} u_0 = 0$, *there exists a unique global solution u to* (NSC) *in* $C([0,\infty),\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))^3 \cap L^4(0,\infty;\dot{H}^{\frac{1}{2}}_3(\mathbb{R}^3))^3$.

Remark 1.5. The space $L^4(0,\infty;\dot{H}^{\frac{1}{2}}_3(\mathbb{R}^3))$ in Theorem 1.4 is scaling invariant space in the case $\Omega = 0$ since $\theta_0 = 4$, $s_0 = 1/2$ and $p_0 = 3$ satisfy (1.4).

Remark 1.6. For the original Navier-Stokes equations

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u + (u\cdot\nabla)u + \nabla p = 0 & t > 0,\ x \in \mathbb{R}^3, \\ \operatorname{div}\, u = 0 & t > 0,\ x \in \mathbb{R}^3, \\ u(0,x) = u_0(x) & x \in \mathbb{R}^3, \end{cases} \tag{NS}$$

it is known by the results of Brezis [5], Giga [9] and Kozono [16] that the existence time T of local solutions for initial data in $L^r(\mathbb{R}^3)$ $(3 < r < \infty)$ and $L^3(\mathbb{R}^3)$ is determined by the each bounded set B in $L^r(\mathbb{R}^3)$ $(3 < r < \infty)$ and the each precompact set K in $L^3(\mathbb{R}^3)$, respectively. Note that the space $L^3(\mathbb{R}^3)$ is a scaling critical space to (NS). On the other hand, the sufficint speed Ω to obtain global solutions is determined by the bounded sets and precompact sets in Theorem 1.1 and Theorem 1.4, respectively. Therefore, our theorems can be regarded as a counterpart of such results from the viewpoint of the Coriolis parameter Ω for the existence of global solutions.

In this paper, we prove Theorem 1.1 only. For the proof of Theorem 1.4, see [14]. In Section 2, we introduce propositions to prove Theorem 1.1 which are on linear estimates for the semigroup $T_\Omega(\cdot)$ and the bilinear estimate. In Section 3, we prove Theorem 1.1.

2. Preliminaries

In what follows, we denote by $C > 0$ various constants and by $0 < c < 1$ various small constants. In order to introduce propositions to prove theorems, let us recall the definition of the homogeneous Besov spaces in brief. Let ϕ be a radial smooth function satisfying

$$\operatorname{supp}\widehat{\phi} \subset \{\,\xi \in \mathbb{R}^3 \mid 2^{-1} \le |\xi| \le 2\,\}, \quad \sum_{j\in\mathbb{Z}} \widehat{\phi}(2^{-j}\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^3 \setminus \{0\}.$$

Let $\{\phi_j\}_{j\in\mathbb{Z}}$ be defined by

$$\phi_j(x) := 2^{3j}\phi(2^j x) \quad \text{for } j \in \mathbb{Z}, x \in \mathbb{R}^3.$$

Then, for $s \in \mathbb{R}, 1 \le p, q \le \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$ is defined by the set of all tempered distributions $f \in \mathcal{S}'(\mathbb{R}^3)$ with

$$\|f\|_{\dot{B}^s_{p,q}} := \left\|\left\{2^{sj}\|\phi_j * f\|_{L^p(\mathbb{R}^3)}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q(\mathbb{Z})} < \infty.$$

Lemma 2.1. [13] *Let $2 \le p \le \infty$. There exists $C > 0$ such that*

$$\|\mathcal{F}^{-1}e^{\pm i\frac{\xi_3}{|\xi|}\Omega t}\mathcal{F}f\|_{\dot{B}^0_{p,2}} \le C\left\{\frac{\log(e+|\Omega|t)}{1+|\Omega|t}\right\}^{\frac{1}{2}(1-\frac{2}{p})}\|f\|_{\dot{B}^{3(1-\frac{2}{p})}_{\frac{p}{p-1},2}} \tag{2.1}$$

for all $\Omega \in \mathbb{R}$, $t > 0$, $f \in \dot{B}^{3(1-\frac{2}{p})}_{\frac{p}{p-1},2}(\mathbb{R}^3)$.

Lemma 2.2. *Let $1 < q \le 2 \le p < \infty$ satisfy $1/q \ge 1 - 1/p$. Then, there exists $C > 0$ such that*

$$\|T_\Omega(t)f\|_{L^p} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\left\{\frac{\log(e+|\Omega|t)}{1+|\Omega|t}\right\}^{\frac{1}{2}(1-\frac{2}{p})}\|f\|_{L^q} \tag{2.2}$$

for all $\Omega \in \mathbb{R}, t > 0, f \in L^q(\mathbb{R}^3)$.

Proof. By the continuous embedding $\dot{B}^0_{p,2}(\mathbb{R}^3) \hookrightarrow L^p(\mathbb{R}^3)$ and (2.1), we have

$$\|T_\Omega(t)f\|_{L^p} \le C\|T_\Omega(t)f\|_{\dot{B}^0_{p,2}} \le C\left\{\frac{\log(e+|\Omega|t)}{1+|\Omega|t}\right\}^{\frac{1}{2}(1-\frac{2}{p})}\|e^{t\Delta}f\|_{\dot{B}^{3(1-\frac{2}{p})}_{\frac{p}{p-1},2}}.$$

And we have from Lemma 2.2 in [17] and the continuous embedding $L^q(\mathbb{R}^3) \hookrightarrow \dot{B}^0_{q,2}(\mathbb{R}^3)$

$$\|e^{t\Delta}f\|_{\dot{B}^{3(1-\frac{2}{p})}_{\frac{p}{p-1},2}} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{\dot{B}^0_{q,2}} \le Ct^{-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q(\mathbb{R}^3)}.$$

Therefore, we obtain (2.2). □

Proposition 2.3. [13] *Let $2 < p < 6$, $2 < \theta < \infty$ satisfy*

$$\frac{3}{4} - \frac{3}{2p} \le \frac{1}{\theta} < 1 - \frac{2}{p}.$$

Then, there exists $C > 0$ such that

$$\|T_\Omega(\cdot)f\|_{L^\theta(0,\infty;L^p)} \le C|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|f\|_{L^2}$$

for all $\Omega \in \mathbb{R}\setminus\{0\}$, $f \in L^2(\mathbb{R}^3)$.

Proposition 2.4. [14] *For every $f \in \dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$, it holds that*

$$\lim_{|\Omega|\to\infty}\|T_\Omega(\cdot)f\|_{L^4(0,\infty;\dot{H}^{\frac{1}{3}}_3)} = 0. \tag{2.3}$$

Remark 2.5. The space $L^4(0,\infty;\dot{H}^{\frac{1}{2}}(\mathbb{R}^3))$ is scaling invariant function space to (NSC) with $\Omega = 0$. (2.3) is proved by Proposition 2.3 and the approximation of functions in $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ by smooth functions. (2.3) is used for the proof of Theorem 1.4.

Proposition 2.6. *Let $2 < p < 3$ and $6/5 < q < 2$ satisfy*

$$1 - \frac{1}{p} \le \frac{1}{q} < \frac{1}{3} + \frac{1}{p}, \tag{2.4}$$

$$\max\left\{0, \frac{1}{2} - \frac{3}{2}\Big(\frac{1}{q} - \frac{1}{p}\Big) - \frac{1}{2}\Big(1 - \frac{2}{p}\Big)\right\} < \frac{1}{\theta} \le \frac{1}{2} - \frac{3}{2}\Big(\frac{1}{q} - \frac{1}{p}\Big). \tag{2.5}$$

Then, there exists $C > 0$ such that

$$\Big\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{L^\theta(0,\infty;\dot H^s_p)} \le C|\Omega|^{-\{\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{\theta}\}}\|f\|_{L^{\frac{\theta}{2}}(0,\infty;\dot H^s_q)} \tag{2.6}$$

for all $s \in \mathbb{R}$, $\Omega \in \mathbb{R}\setminus\{0\}, f \in L^{\frac{\theta}{2}}(0,\infty;\dot H^s_q(\mathbb{R}^3))$.

Proof. We only consider the case $s = 0$ for simplicity since the case $s \neq 0$ is treated similarly. By Lemma 2.2, we have

$$\begin{aligned}
&\Big\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla f(\tau)d\tau \Big\|_{L^\theta(0,\infty;L^p)} \\
&\le C\Big\| \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\Big\{\frac{\log(e+|\Omega||t-\tau|)}{1+|\Omega||t-\tau|}\Big\}^{\frac{1}{2}(1-\frac{2}{p})}\|f(\tau)\|_{L^q}d\tau\Big\|_{L^\theta(0,\infty)}.
\end{aligned}$$

In the case $1/\theta = 1/2 - 3(1/q-1/p)/2$, we have from Hardy-Littlewood-Sobolev's inequality

$$\begin{aligned}
&\Big\| \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\Big\{\frac{\log(e+|\Omega||t-\tau|)}{1+|\Omega||t-\tau|}\Big\}^{\frac{1}{2}(1-\frac{2}{p})}\|f(\tau)\|_{L^q}d\tau\Big\|_{L^\theta(0,\infty)} \\
&\le \Big\| \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|f(\tau)\|_{L^q}d\tau\Big\|_{L^\theta(0,\infty)} \\
&\le C\|f\|_{L^{\frac{\theta}{2}}(0,\infty;L^q)}.
\end{aligned}$$

In the case $1/\theta < 1/2 - 3(1/q-1/p)/2$, we have from Hausdorff-Young's inequality with $1/\theta = 2/\theta + 1/r - 1$

$$\begin{aligned}
&\Big\| \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\Big\{\frac{\log(e+|\Omega||t-\tau|)}{1+|\Omega||t-\tau|}\Big\}^{\frac{1}{2}(1-\frac{2}{p})}\|f(\tau)\|_{L^q}d\tau\Big\|_{L^\theta(0,\infty)} \\
&\le \Big\| t^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\Big\{\frac{\log(e+|\Omega|t)}{1+|\Omega|t}\Big\}^{\frac{1}{2}(1-\frac{2}{p})}\Big\|_{L^r(0,\infty)}\|f\|_{L^{\frac{\theta}{2}}(0,\infty;L^q)} \\
&= C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^{\frac{\theta}{2}}(0,\infty;L^q)}.
\end{aligned}$$

Therefore, we obtain (2.6). □

Lemma 2.7. *Let s, p satisfy*

$$0 \le s < 3, \quad \frac{s}{3} < \frac{1}{p} < \frac{1}{2} + \frac{s}{6},$$

and let q satisfy

$$\frac{1}{q} = \frac{2}{p} - \frac{s}{3}.$$

Then, there exists $C > 0$ such that

$$\|fg\|_{\dot H^s_q} \le C\|f\|_{\dot H^s_p}\|g\|_{\dot H^s_p}. \tag{2.7}$$

Proof. Let r satisfy $1/q = 1/p + 1/r$. In the Sobolev spaces, it is known that

$$\|fg\|_{\dot H^s_q} \le C\|f\|_{\dot H^s_p}\|g\|_{L^r} + C\|f\|_{L^r}\|g\|_{\dot H^s_p}.$$

By the continuous embedding $\dot H^s_p(\mathbb{R}^3) \hookrightarrow L^r(\mathbb{R}^3)$, we obtain (2.7). □

3. Proof of Theorem 1.1

Since the assumption on θ and p in Proposition 2.3 is satisfied by (1.1) and (1.2), there exists $C_0 > 0$ such that

$$\|T_\Omega(\cdot)u_0\|_{L^\theta(0,\infty;\dot H^s_p)} \le |\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}C_0\|u_0\|_{\dot H^s}.$$

Let $\Psi(u)$ and Y be defined by

$$\Psi(u)(t) := T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)(\tau)d\tau, \tag{3.1}$$

$$Y := \{\, u \in L^\theta(0,\infty;\dot H^s_p(\mathbb{R}^3))^3 \mid \|u\|_{L^\theta(0,\infty;\dot H^s_p)} \le 2C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot H^s}, \operatorname{div} u = 0 \,\},$$
$$d(u,v) := \|u-v\|_{L^\theta(0,\infty;\dot H^s_p)}.$$

Let q satisfy $1/q = 2/p - s/3$. Since the assumptions on s, p, q and θ in Proposition 2.6 and Lemma 2.7 are satisfied by (1.1) and (1.2), for any $u, v \in Y$, we have from Proposition 2.3, Proposition 2.6 and Lemma 2.7

$$\begin{aligned}
\|\Psi(u)\|_{L^\theta(0,\infty;\dot H^s_p)} &\le C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot H^s} + C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|u\otimes u\|_{L^{\frac{\theta}{2}}(0,\infty;\dot H^s_q)}\\
&\le C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot H^s} + C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|u\|^2_{L^\theta(0,\infty;\dot H^s_p)}\\
&\le C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot H^s} + C_1|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})+2\{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})\}}\|u_0\|^2_{\dot H^s}\\
&\le C_0|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot H^s} + C_1|\Omega|^{-\frac{s}{2}+\frac{1}{4}}|\Omega|^{-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|^2_{\dot H^s},
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\|\Psi(u)-\Psi(v)\|_{L^\theta(0,\infty;\dot H^s_p)}\\
&= \left\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot\{u\otimes(u-v)(\tau) + (u-v)\otimes v(\tau)\}d\tau \right\|_{L^\theta(0,\infty;\dot H^s_p)}\\
&\le C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\|u\otimes(u-v)+(u-v)\otimes v\|_{L^{\frac{\theta}{2}}(0,\infty;\dot H^s_q)}\\
&\le C|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})}\big(\|u\|_{L^\theta(0,\infty;\dot H^s_p)} + \|v\|_{L^\theta(0,\infty;\dot H^s_p)}\big)\|u-v\|_{L^\theta(0,\infty;\dot H^s_p)}\\
&\le C_2|\Omega|^{\frac{1}{\theta}-\frac{1}{2}+\frac{3}{2}(\frac{1}{q}-\frac{1}{p})-\frac{1}{\theta}+\frac{3}{4}(1-\frac{2}{p})}\|u_0\|_{\dot H^s}\|u-v\|_{L^\theta(0,\infty;\dot H^s_p)}\\
&= C_2|\Omega|^{\frac{1}{4}+\frac{3}{2q}-\frac{3}{p}}\|u_0\|_{\dot H^s}\|u-v\|_{L^\theta(0,\infty;\dot H^s_p)}\\
&= C_2|\Omega|^{-\frac{s}{2}+\frac{1}{4}}\|u_0\|_{\dot H^s}\|u-v\|_{L^\theta(0,\infty;\dot H^s_p)}.
\end{aligned}$$

If Ω, u_0 satisfy

$$C_1|\Omega|^{-\frac{s}{2}+\frac{1}{4}}\|u_0\|_{\dot H^s} \le C_0, \quad C_2|\Omega|^{-\frac{s}{2}+\frac{1}{4}}\|u_0\|_{\dot H^s} \le \frac{1}{2},$$

then, it is possible to apply Banach's fixed point theorem in Y and we obtain $u \in Y$ with

$$u(t) = T_\Omega(t)u_0 - \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla\cdot(u\otimes u)d\tau.$$

Here, we show that the solution $u \in Y$ satisfies $u(t) \in \dot{H}^s(\mathbb{R}^3)^3$ for all $t \geq 0$. On the linear part, it is easy to see that $T_\Omega(t)u_0 \in \dot{H}^s(\mathbb{R}^3)^3$ for any $t \geq 0$. On the nonlinear part, let $1/q = 2/p - s/3$ and we have from Lemma 2.2, Lemma 2.7 and Hölder's inequality

$$\begin{aligned}
\left\| \int_0^t T_\Omega(t-\tau)\mathbb{P}\nabla \cdot (u \otimes u)(\tau)d\tau \right\|_{\dot{H}^s} &\leq C \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})} \|(u \otimes u)(\tau)\|_{\dot{H}^s_q} d\tau \\
&\leq C \int_0^t (t-\tau)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})} \|u(\tau)\|^2_{\dot{H}^s_p} d\tau \\
&\leq C \left\| (t-\cdot)^{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})} \right\|_{L^{\frac{\theta}{\theta-2}}(0<\tau<t)} \left\| \|u(\tau)\|^2_{\dot{H}^s_p} \right\|_{L^{\frac{\theta}{2}}(0,\infty)} \\
&\leq C t^{\frac{\theta-2}{\theta}[1-\frac{\theta}{\theta-2}\{-\frac{1}{2}-\frac{3}{2}(\frac{1}{q}-\frac{1}{2})\}]} \|u\|^2_{L^\theta(0,\infty;\dot{H}^s_p)}.
\end{aligned} \tag{3.3}$$

Here, we note on the integrability at $\tau = t$ that

$$\frac{\theta}{\theta-2}\left\{\frac{1}{2} + \frac{3}{2}\left(\frac{1}{q} - \frac{1}{2}\right)\right\} < 1 \quad \text{if and only if} \quad \frac{1}{\theta} < \frac{5}{8} - \frac{3}{2p} + \frac{s}{4}.$$

Therefore, we obtain $u(t) \in \dot{H}^s(\mathbb{R}^3)^3$ and we also see $u \in C([0,\infty), \dot{H}^s(\mathbb{R}^3))^3$. □

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